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Math 2934

exm 3 review

Question 1. $\int$ between regions:

1) $\int\int_D 2yx^2 + 9y^3 dA$, where R is the region between $y = \frac{2}{3}x \quad y = 2\sqrt{x}$
   - **Find bounds:** Set regions equal to each other, result in:

$$x = [0, 9] \text{ outer bounds}$$

$$y = [\frac{2}{3}x, 2\sqrt{x}] \text{ inner bounds}$$

   - Put $\int$ tgt

$$\int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3 \; dy \; dx$$

2) $\int\int_R xy^2 \, dA$ where R is the region between $x = y^2$ and $x = 2 - 2y^2$
   - **Bounds:**

$$y^2 = 2 - 2y^2 \quad \rightarrow y = [\pm\sqrt{\frac{2}{3}}]$$

$$x = [y^2, 2 - 2y^2]$$

   - put tgt (bounds dependent on what u solve for first, x or y first?

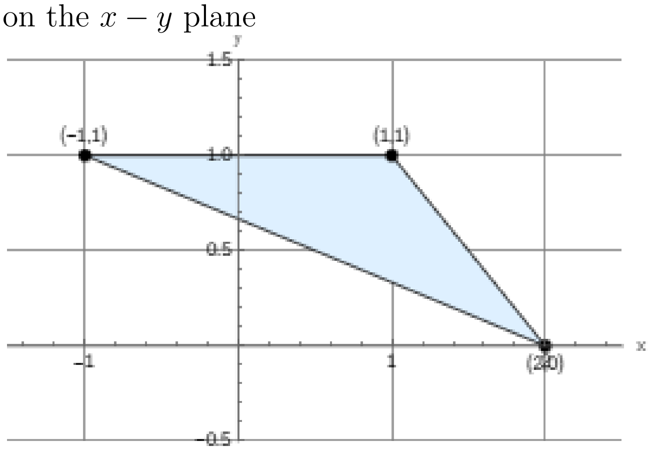
$$\int_{-\sqrt{\frac{2}{3}}}^{\sqrt{\frac{2}{3}}} \int_{2-2y^2}^{y^2} xy^2 \; dx \; dy$$

Question 2. $\int$ surface area of a region:
Formula:

$$S = \int\int_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2} dA$$

Bounds found w/ graph

1) Σ: the region on the graph of $g(x, y) = x^2 + y^2$ that lies above the triangle with vertices $(1, 1), (-1, 1), (2, 0)$ on the $x - y$ plane



- Create 3 lines to make triangle and find bounds using point-slope formula of $y - y_1 = m(x - x_1)$
  First line:

$$\text{for (1,1), (-1,1): } \frac{1-1}{-1-1} = 0,\ y = 1$$

$$\text{for (1,1), (2,0): } \frac{0-1}{2-1} = \frac{-1}{1},\ y - 1 = -(x-1),\ y = -x + 2$$

$$\text{for (-1,1) and (2,0): } \frac{0-1}{2--1} = -\frac{1}{3},\ y = -\frac{1}{3}x + \frac{2}{3}$$

- Get bounds from all 3 points. In this case, hv 2 max y-heights and we split $\int$ at $x = 1$.
- Convert inside, just find both partials

2) Σ: the region on the graph of $g(x, y) = x + y$ that lies above the disk with radius 2 centered on the origin.

   Know disk is

$$x^2 + y^2 = 2$$

$$r = 2$$

$$\theta = 2\pi$$

   and we $\int$ around some $dr\ d\theta$

3) Σ: the region on the graph of $g(x, y) = x^2 + 1$ that lies above the annulus with inner radius 1 and outer radius 2 centered on the origin. a disk (or annulus) centered at the origin with radius r

$$\text{inner} \int \to r_{\text{inner}} \leq \sqrt{x^2 + y^2} \leq r_{\text{outer}}$$

$$\text{outer} \int \to [0, 2\pi]$$

Question 3. Evaluate the following integrals:

1) $\iint_R 2x^2 + y^2\, dA$, R is the pie wedge given by $0 \leq \theta \leq \pi/3$ and $0 \leq r \leq 2$

   polar w/ jacobian + transformation (bounds stay samey :)) We given bounds alrdy

2) $\iint_R \frac{1}{x^2+y^2+1}\, dA$, where R is the annulus centered on the origin with inner radius 2 and outer radius 4.

   polar transformation, see 2.3 for bounds

3) $\iint_R \sqrt{x^2 + y^2}\, dA$ where R is the unit disk. Bounds:

$$\int_0^{2\pi} \int_0^1 ...dA$$

Question 4. Evaluate the following cylindrical integrals:

1) $\iiint_D x^2 + y^2 + z^2 \, dV$ where D is the cylindrical shell with inner radius 1, outer radius 2, which is bounded by $z = 1$ and $z = 4$

   Cylindrical Cords review:

$$\text{Transformation: } x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \quad x^2 + y^2 + z^2 = r^2 + z^2$$

$$\text{Jacobian: } dV = r \, dz \, dr \, d\theta$$

$$LOI: \; r = [1, 2] \; \varphi = [0, 2\pi] \; z = [1, 4]$$

   Then transform inside, place integration bounds, add jacobian solve

2) $\iiint_D x + y \, dV$ where D is the region represented by $[0, 1] \times [\pi/3, \pi/4] \times [-1, 1]$ in $r - \theta - z$ space.
   Still cylindrical cords, though need to negate outer $\int$ given $-[\pi/4, \pi/3]$ to correct bounds.

Question 5. Evaluate the following spherical integrals: Recall spherical stuff:

$$\text{conversion factors: } \begin{matrix} x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi \\ x^2 + y^2 + z^2 = \rho^2 \end{matrix}$$

$$\text{Jacobian: } dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$\text{plug into: } \iiint\limits_E f(x, y, z) \; dV$$

1) $\iiint_D x^2 + y^2 \, dV$, where D is the volume given by the inequalities $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$
   LOI (hard part) know:
   - radius $= [0, 1]$
   - Azimuthal/θ spans full region, so its $[0, 2\pi]$
   - z is bounded by $z \geq 0 \to [0, \pi/2]$

2) $\iiint_D x - y^2 \, dV$, where D is the spherical shell with inner radius 1 and outer radius 2
   LOI
   - $\rho = [1, 2]$
   - $\varphi = [0, \pi]$
   - $\theta = [0, 2\pi]$

3) $\iiint_D e^{x+y+z} \, dV$ where D is the cone given by $\varphi \leq \pi/4$

$$\rho = [0, 1]$$

$$\varphi = [0, \pi/4]$$

$$\theta = [0, 2\pi]$$

4) $\iiint_D \frac{r}{x} \, dV$ where D is the lime wedge given by $-\pi/3 \leq \theta \leq \pi/3$

$$\theta = [-\frac{\pi}{3}]$$

$$r = [0, 1]$$

$$z = [0, 1]$$

Question 6. Set up the following spherical surface areas $\int$:

1) Consider the sphere of radius 3 and let Σ be the cap made by $\varphi \leq \pi/3$. Set up the integral which gives you the surface area of Σ Recall spherical stuff (in this case no use formula above)

$$\text{Formula: } \int\int_D dS$$

$$dS = R^2 sin\varphi \; d\varphi \; d\theta$$

$$LOI : r = 3 \; \varphi = [0, \pi/3] \; \theta = [0, 2\pi]$$

$$\int_0^{2\pi} \int_0^{\pi/3} 3^2 sin\varphi \; d\varphi \; d\theta$$

2) Consider the earth modeled as a sphere of radius 3963 miles, and let Σ be the tropics, which are defined to be all the points on the surface of the earth between 23.5 degrees above the equator and 23.5 degrees below the equator. Set up the integral that would give you however many square miles are in the tropics (of both sea and land). (Note: you will need to convert everything to radians for the calculus to work out) Same general formula,

Question 7. Let $S(u, v) = \langle u, u^2 + v \rangle$, and let $R_{uv} = [0, 1] \times [0, 2]$. Plot $R = S(R_{uv})$ on the $x - y$ plane. What is the area of R?

- Find transformation to map $S(u, v)$ in this case, $x = \hat{\imath} \quad y = \hat{\jmath}$
- We use x-bounded limits $\int_0^1 \hat{\jmath} \; dx$, to find y, simply plug in $[0, 2]$ for $\hat{\jmath}$.

$$\int_0^1 2dx$$

To plot:

- Given only $\hat{\imath}$ is constant, we can simply push $[0, 1] = u$ to obtain 2 points;
- $\hat{\jmath}$ obtains 2 lines by pushing $[0, 2] = v$, obtain 2 more pts $\rightarrow$ 4 pts total

Question 8. For each of the following, what are the dimensions of the described objects and what are the dimensions of the spaces these objects are contained within. **Write down the notation for the corresponding integral which gives you the arc-length/surface area/volume of the region.** (You don't have to evaluate, you just need to write the correct notation)

Surface Area (where dA is $dx\;dy$ or whatever):

$$S = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1} \, dA$$

Parametric form ($\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle$): The surface area formula is:

$$S = \iint_D \|\mathbf{r}_s \times \mathbf{r}_t\| \, ds \, dt,$$

where $\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial s}$ and $\mathbf{r}_t = \frac{\partial \mathbf{r}}{\partial t}$.

- $S(s) =< s, s + 1 >$
- LHS = dim of object
- RHS = dim of space
- So more commas in RHS ++ space
- more vars LHS ++ object

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- no. of [...] $\times$ [...] $\times$ ... = #d object
- number of args in $g(...)$ +1 = dimension of space

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- Say we have $x^2 + y^2 = 4$: 1D object in 2D space
- $x^2 + y^2 = 4 \quad x - y = 4$ -> 0D object in 2D space
- $x^2 + y^2 + z^2 = 1$? 2D in 3D space!

1) The region on the graph of $g(x, y) = \sin(xy)$ which lies above $[0, 1] \times [1, 2]$ on the $x - y$ plane
2) The inequality $x^2 + y^2 - z^2 \geq 1$

   3d (xyz args) dimension of object, 3d space $no.var - 1(+1)$, whr +1 from $\geq$

   Because $\geq$ results in unbounded region, it's ∞
3) The equality $2x^2 + y^2 + z^2 = 1$

   3d dim (xyz args), 2d object frm $no.var - 1$
4) The set of solutions to the system of equations: $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 = z$

   3d object, 2d space (same as above)

5) The region parameterized by $S(s, t) = \langle s, s + t, s^2 + t^2 \rangle$ for $0 \leq s \leq 1$ and $1 \leq t \leq 3$

   2d object, 3d space

   Follow **Parametric form rules**, $\int$ bounds given alrdy. (Work this out urself later)
6) The region parameterized by $r(t) = \langle t, t^2, t^3, t^4 \rangle$

   1d object, 4d space

   arc-len formula

$$\int_a^b ||r'(t)||dt$$

Question 9. Line $\int$'s:

Polar cords stuff (whr dA = jacobian):

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

$$dA = r\,dr\,d\theta$$

$$\iint_D f(x, y)\,dA$$

Line Integral stuff:

- Map transformation formula to inside using

$$\vec{r}(t) = (1 - t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle \quad , \ 0 \leq t \leq 1$$

or

$$\begin{aligned} x &= (1 - t)\,x_0 + t\,x_1 \\ y &= (1 - t)\,y_0 + t\,y_1 \quad 0 \leq t \leq 1 \\ z &= (1 - t)\,z_0 + t\,z_1 \end{aligned}$$

- Break up into separate lines, and individually integrate then add up once finished
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$$\int_a^b f\left(h\left(t\right), g\left(t\right)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1) $\int_{\partial R} x^2 - y^2 \, ds$ where R is the region given by $x^2 + y^2 \leq 1$ B/c given circle ! need 2 split $\int$

- **Parameterization** of circle, in this case $x = cos(t) \quad y = sin(t)$ given its equal to 1
- **Range**: $-\pi/2 \leq t \leq \pi/2$
- Calculate

$$||r'(t)|| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- $\frac{dx}{dt} = sin^2 \quad \frac{dy}{dt} = cos^2$ (aft transformation)

$$\int_0^{2\pi} cos^2 - sin^2 d\theta$$

2) $\int_{\partial R} x^2 + y \, ds$ where R is the triangle with vertices $(1, 1)$, $(1, -1)$, $(0, -1)$

We have 3 lines, whr $f(x, y) = x^2 + y$:

$$\int_{L_{1,2}} f(x, y)\, ds + \int_{L_{2,3}} f(x, y)\, ds + \ldots$$

$$L_{1,2} = \begin{matrix} x(t)_{1,2} = (1-t)\cdot 1 + t \cdot 1 = 1, \\ y(t)_{1,2} = (1-t)\cdot 1 + t\cdot(-1) = 1 - 2t, \end{matrix} \rightarrow 1^2 + (1 - 2t) = \boxed{2 - 2t = L_{1,2}} \text{ transformation}$$

$$L_{2,3} = \ldots$$

$$L_{3,1} = \ldots$$

After transforming, still need to find $||r'(t)||$, this is given by:

$$\frac{dx(t)_{1,2}}{dt} = \frac{dx}{dt}(1) = 0$$

$$\frac{dy(t)_{1,2}}{dt} = \frac{dy}{dt}(1 - t^2) = -2t$$

Plugging into

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2$$

Putting it all tgt we hv:

$$\int_0^1 (2 - 2t)(2)\ dt\ + \int_a^b L_{2,3}\ ||r'(t)||\ dt\ + \ldots$$

3) $\int_{\partial R} (x - y)^2 \, ds$ where R is the square with vertices $(0, 0)$, $(1, 1)$, $(0, 2)$, $(-1, 1)$

Question 10. For each vector field V, plot V (with at least 9 vectors), check if V is conservative, and give the corresponding potential function if applicable. Find Conservative? $\partial_{\hat{i}_y} = \partial_{\hat{j}_x}$? y : n Use following steps for potentials:

- Find $\int \hat{i}dx = f(x,y)$ (this represents our final eq)
- outputs with some $+h(y)$ as constant. Find by taking $\partial f_y(x,y) = \hat{j}$
- take $\int$ of $h'(y)$ to find $h(y)$, plug into $f(x,y)$ to find answer.

1) $V(x,y) = (x^2 + y, x - y^2)$
   Conservative? Does $\partial\hat{i}_y = \partial\hat{j}_x$? $\rightarrow yes$ Step 1)

$$\int \hat{i}dx = \int x^2 + y\ dx = \frac{x^3}{3} + xy + h(y) = f(x,y)$$

   Find $h(y)$:

$$\partial_y f(x,y) = \partial_y[\frac{x^3}{3} + xy + h(y)] = x + h'(y)$$
$$x + h'(y) = x - y^2 \rightarrow h'(y) = -y^2$$

   Finding $h(y)$

$$\int h'(y)dy = \int -y^2 dy = -\frac{y^3}{3} + C$$

   Put together:

$$\boxed{\frac{x^3}{3} + xy + -\frac{y^3}{3} + C}$$

2) $V(x,y) = (x,y)$
3) $V(x,y) = (y,x)$
4) $V(x,y) = (y,-x)$

Question 11. For each of the following vector fields, find the curl and divergence:

$$\text{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$\text{whr PQR} \rightarrow ijk$$

1) $V(x,y,z) = \langle xy, yz, zx \rangle$
2) $V(x,y,z) = \langle x+y, y+z, z+x \rangle$
3) $V(x,y,z) = \langle (x+z)^2, (y+x)^2, (z+y)^2 \rangle$

Question 12. Things to Know:

- **Spheres:** A sphere of radius r centered at (x_0, y_0, z_0) can be parameterized using spherical coordinates:

$$\vec{r}(\theta, \varphi) = \begin{bmatrix} x_0 + r\sin\varphi\cos\theta \\ y_0 + r\sin\varphi\sin\theta \\ z_0 + r\cos\varphi \end{bmatrix}, \quad 0 \le \theta < 2\pi, \quad 0 \le \varphi \le \pi$$

- **Cylinders:** A circular cylinder of radius r aligned along the z-axis and centered at (x_0, y_0):

$$\vec{r}(\theta, z) = \begin{bmatrix} x_0 + r\cos\theta \\ y_0 + r\sin\theta \\ z \end{bmatrix}, \quad 0 \le \theta < 2\pi, \quad z \in [z_{\text{min}}, z_{\text{max}}]$$

- **Circles:** A circle of radius r in the xy-plane centered at (x_0, y_0):

$$\vec{r}(\theta) = \begin{bmatrix} x_0 + r\cos\theta \\ y_0 + r\sin\theta \end{bmatrix}, \quad 0 \le \theta < 2\pi$$

  Orientation is counter-clockwise for increasing θ.
- **Lines:** A line through point $\vec{r}_0$ in the direction of vector $\vec{v}$:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R}$$

  Orientation is in the direction of $\vec{v}$.
- **Line segments:** A segment from point $\vec{A}$ to point $\vec{B}$:

$$\vec{r}(t) = (1-t)\vec{A} + t\vec{B}, \quad 0 \le t \le 1$$

  Orientation is from $\vec{A}$ to $\vec{B}$ as t increases.
- **Ellipses:** An ellipse centered at (x_0, y_0) with semi-axes a and b:

$$\vec{r}(\theta) = \begin{bmatrix} x_0 + a\cos\theta \\ y_0 + b\sin\theta \end{bmatrix}, \quad 0 \le \theta < 2\pi$$

  Orientation is counter-clockwise.

1) what inequalities look like in cylindrical coordinates
2) What inequalities look like in spherical coordinates

3) trig integrals

Form	Looks Like	Substitution
$\sqrt{b^2x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec\theta$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \frac{a}{b}\sin\theta$
$\sqrt{a^2 + b^2x^2}$	$\tan^2\theta + 1 = \sec^2\theta$	$x = \frac{a}{b}\tan\theta$

4) how to push points, curves, and regions through parameterizations